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A note on cancellation of projective modules

Alpesh M. Dhorajia, Manoj K. Keshari *

Department of Mathematics, IIT Mumbai, Mumbai - 400076, India

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Dedicated to Professor S.M. Bhatwadekar on his 65th birthday

ABSTRACT

We prove the following results. (i) Let A be an affine algebra of dimension $d \geq 4$ over $\overline{\mathbb{F}}_p$ (with $p \geq d$). Then all projective A-modules of rank $d-1$ are cancellative.

(ii) Let A be a ring of dimension d such that $E_{d+1}(R)$ acts transitively on $\mathrm{Um}_{d+1}(R)$ for every finite extension R of A. Then for any projective A-module P of rank d, $E(A \oplus P)$ acts transitively on $\mathrm{Um}(A \oplus P)$.

1. Introduction

All the rings are assumed to be commutative Noetherian and all the modules are finitely generated. Let A be a ring and let P be a projective A-module. We say that P is *cancellative* if $P \oplus A^r \xrightarrow{\sim} P' \oplus A^r$ for some positive integer r and some projective A-module P' implies that $P \xrightarrow{\sim} P'$. A classical result of Bass [2] says that *if rank* $P > \dim A$, *then* $E(A \oplus P)$ *acts transitively on* $\mathrm{Um}(A \oplus P)$. *In particular, P is cancellative.*

Let A be a ring of dimension d and let P be a projective A-module of rank d. It is interesting to know under what conditions A^d is cancellative implies that every projective A-module P of rank d is cancellative. Bhatwadekar ([3], Example 2.11) gave an example of a smooth affine surface A over $\mathbb{R}$ such that A^2 is cancellative but $A \oplus K_A$ is not cancellative, where K_A is the canonical module of A. The second author obtained a sufficient condition ([9], Theorem 3.6) by proving the following result.

Let A be a ring of dimension d. Assume that if R is a finite extension of A then R^d is cancellative. Then every projective A-module P of rank d is also cancellative. In other words, if $\mathrm{GL}_{d+1}(R)$ acts transitively on $\mathrm{Um}_{d+1}(R)$ for every finite extension R of A, then $\mathrm{Aut}(A \oplus P)$ acts transitively on $\mathrm{Um}(A \oplus P)$.

Our first result generalizes above result as follows (3.4).

Theorem 1.1. *Let A be a ring of dimension d and let P be a projective A-module of rank d. Assume that if R is a finite extension of A then $E_{d+1}(R)$ acts transitively on* $\mathrm{Um}_{d+1}(R)$. *Then $E(A \oplus P)$ acts transitively on* $\mathrm{Um}(A \oplus P)$.

If A is an affine algebra of dimension d over $\mathbb{Z}$ then Vaserstein [14] proved that $E_{d+1}(A)$ acts transitively on $\mathrm{Um}_{d+1}(A)$. As a consequence of (1.1), we get another proof of the following result of Mohan Kumar et al. ([10], Theorem 2.4) that if P is a projective A-module of rank d, then $E(A \oplus P)$ acts transitively on $\mathrm{Um}(A \oplus P)$.

Let A be a smooth affine algebra of dimension d over an algebraically closed field $\overline{k}$. Assume that gcd $((d-1)!, \mathit{char}(\overline{k})) = 1$. Then Fasel et al. ([5], Theorem 7.3) proved that *stably free A-modules of rank $d-1$ are free*, thus answering an old question of Suslin. In fact, for the case $d \geq 4$, they proved that, A being normal, suffices. In view of their result, a natural question arises: Let P be a projective A-module of rank $d-1$. Is P cancellative? We answer this question in affirmative when $\overline{k} = \overline{\mathbb{F}}_p$. More precisely, we prove the following result (3.5).

* Corresponding author.
E-mail addresses: alpesh@math.iitb.ac.in (A.M. Dhorajia), keshari@math.iitb.ac.in (M.K. Keshari).

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Theorem 1.2. *Let A be an affine algebra of dimension $d \geq 4$ over $\overline{\mathbb{F}}_p$, where $p \geq d$. Then every projective A-module of rank $d-1$ is cancellative.*

Finally, as a consequence of the techniques developed for (1.1), we will prove the following result (4.5). Gubeladze proved this result ([7,8]) in case P is free.

Theorem 1.3. *Let $M \subset \mathbb{Q}^r_+$ be a seminormal monoid such that $M \subset \mathbb{Q}^r_+$ is an integral extension. Let R be a ring of dimension d and let P be a projective $R[M]$-module of rank n. Then $E(R[M] \oplus P)$ acts transitively on* $\mathrm{Um}(R[M] \oplus P)$ *whenever $n \geq$ max $(2, d+1)$.*

2. Preliminaries

Let A be a ring and let M be an A-module. We say that $m \in M$ is *unimodular* if there exists $\phi \in M^* = \mathrm{Hom}_A(M, A)$ such that $\phi(m) = 1$. The set of all unimodular elements of M will be denoted by $\mathrm{Um}(M)$. We denote by $\mathrm{Aut}_A(M)$, the group of all A-automorphism of M.

For an ideal J of A, we denote by $EL^1(A \oplus M, J)$, the subgroup of $\mathrm{Aut}_A(A \oplus M)$ generated by all the automorphisms $\Delta_{a\varphi} = \begin{pmatrix} 1 & a\varphi \\ 0 & id_M \end{pmatrix}$ and $\Gamma_m = \begin{pmatrix} 1 & 0 \\ m & id_M \end{pmatrix}$ with $a \in J$, $\varphi \in M^*$ and $m \in M$. In particular, we denote by $E^1_{r+1}(A, J)$, the subgroup of $E_{r+1}(A)$ generated by $\Delta_{\mathbf{a}} = \begin{pmatrix} 1 & \mathbf{a} \\ 0 & id_F \end{pmatrix}$ and $\Gamma_{\mathbf{b}} = \begin{pmatrix} 1 & 0 \\ \mathbf{b}^t & id_F \end{pmatrix}$, where $F = A^r$, $\mathbf{a} \in JF$ and $\mathbf{b} \in F$. Further, we will write $EL^1(A \oplus M)$ for $EL^1(A \oplus M, A)$.

We denote by $\mathrm{Um}(A \oplus M, J)$ the set of all $(a, m) \in \mathrm{Um}(A \oplus M)$ with $a \in 1 + J$ and $m \in JM$. We will write $\mathrm{Um}_r(A, J)$ for $\mathrm{Um}(A \oplus A^{r-1}, J)$.

Let $p \in M$ and $\varphi \in M^*$ be such that $\varphi(m) = 0$. Let $\varphi_p \in \mathrm{End}(M)$ be defined as $\varphi_p(q) = \varphi(q)p$. Then $1 + \varphi_p$ is a (unipotent) automorphism of M. An automorphism of M of the form $1 + \varphi_p$ is called a *transvection* of M if either $p \in \mathrm{Um}(M)$ or $\varphi \in \mathrm{Um}(M^*)$. We denote by $E(M)$, the subgroup of $\mathrm{Aut}(M)$ generated by all the transvections of M.

The following result is due to Bak et al. ([1], Theorem 3.10). In [4], we proved results for $EL^1(A \oplus P)$. Due to this result, we can interchange $E(A \oplus P)$ and $EL^1(A \oplus P)$.

Theorem 2.1. *Let A be a ring and let P be a projective A-module of rank ≥ 2. Then $EL^1(A \oplus P) = E(A \oplus P)$.*

Remark 2.2. Using (2.1), it is easy to see that if I is any ideal of A, then the natural map $E(A \oplus P) \to E((A \oplus P)/I(A \oplus P))$ is surjective.

The following result is due to Lindel ([11], Lemma 1.1).

Lemma 2.3. *Let A be a ring and let P be a projective A-module of rank r. Then there exists $s \in A$ such that the following holds:*

(i) *P_s is free,*
(ii) *there exists $p_1, \ldots, p_r \in P$ and $\phi_1, \ldots, \phi_r \in \mathrm{Hom}(P, A)$ such that $(\phi_i(p_j)) =$ diagonal $(s, \ldots, s)$,*
(iii) *$sP \subset p_1A + \cdots + p_rA$,*
(iv) *the image of s in A_{red} is a non-zerodivisor and*
(v) *$(0 : sA) = (0 : s^2A)$.*

The following two results are from ([4], Lemma 3.1 and Lemma 3.10).

Lemma 2.4. *Let A be a ring and let P be a projective A-module. Let "bar" denote reduction modulo the nil radical of A. If $E(\overline{A} \oplus \overline{P})$ acts transitively on* $\mathrm{Um}(\overline{A} \oplus \overline{P})$*, then $E(A \oplus P)$ acts transitively on* $\mathrm{Um}(A \oplus P)$.

Lemma 2.5. *Let A be a ring and let P be a projective A-module of rank r. Choose $s \in A$, $p_1, \ldots, p_r \in P$ and $\varphi_1, \ldots, \varphi_r \in P^*$ satisfying the properties of (2.3). Let $(a, p) \in \mathrm{Um}(A \oplus P, sA)$ with $p = c_1p_1 + \cdots + c_rp_r$, where $c_i \in sA$ for $i = 1, \ldots, r$. Assume there exists $\phi \in E^1_{r+1}(A, sA)$ such that $\phi(a, c_1, \ldots, c_r) = (1, 0, \ldots, 0)$. Then there exists $\Phi \in E(A \oplus P)$ such that $\Phi(a, p) = (1, 0)$. (In fact, we get a map from $E^1_{r+1}(A, sA)$ to $E(A \oplus P)$.)*

When A is an affine algebra of dimension d over an algebraically closed field $\overline{k}$, then Suslin [13] proved that Bass cancellation theorem [2] can be strengthened as follows: *If P is a projective A-module of rank d, then* $\mathrm{Aut}(A \oplus P)$ *acts transitively on* $\mathrm{Um}(A \oplus P)$, *i.e. P is cancellative*. Mohan Kumar et al. ([10], Theorem 2.4) generalized Suslin's result in case $\overline{k} = \overline{\mathbb{F}}_p$ as follows.

Theorem 2.6. *Let A be an affine algebra of dimension $d \geq 2$ over $\overline{\mathbb{F}}_p$. Let P be a projective A-module of rank d. Then $E(A \oplus P)$ acts transitively on* $\mathrm{Um}(A \oplus P)$.

The following result ([9], Theorem 3.8) is very crucial for the proof of (3.5).

Theorem 2.7. *Let A be an affine algebra of dimension d over $\overline{\mathbb{F}}_p$. Assume that if R is a finite extension of A then R^{d-1} is cancellative. Then every projective A-module P of rank $d-1$ is cancellative.*

We end this section with a result due to Fasel et al. ([5], Corollary 7.4).

Proposition 2.8. *Let R be an affine algebra of dimension $d \geq 4$ over an algebraically closed field $\overline{k}$. Assume that gcd $((d-1)!, char(k)) = 1$. Let J be the ideal defining the singular locus of R. Then for any $v \in \mathrm{Um}_d(R, J)$, there exists $\Theta \in \mathrm{GL}_d(R)$ such that $v\Theta = (1, 0, \ldots, 0)$.*

3. Main theorem

In this section, we prove our main result.

Let A be a ring and I an ideal of A. For an integer $n \geq 3$, define $E_n(I)$ as the subgroup of $E_n(A)$ generated by $E_{ij}(a) = Id + ae_{ij}$, where $a \in I$, $1 \leq i \neq j \leq n$ and only non-zero entry of the matrix ae_{ij} is a at the (i,j)th place.

Consider the cartesian square

$$\begin{tikzcd} A(I) \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "j_1"] \\ A \arrow[r, "j_2"'] & A/I \end{tikzcd}$$

The relative group $E_n(A, I)$ is defined in [12] by the exact sequence

$$1 \to E_n(A, I) \to E_n(A(I)) \xrightarrow{E_n(p_1)} E_n(A) \to 1$$

and it is shown ([12], Proposition 2.2) that $E_n(A, I)$ is isomorphic to the kernel of the natural map $E_n(A) \to E_n(A/I)$. Further, $E_n(A, I)$ is the normal closure of $E_n(I)$ in $E_n(A)$ ([13], Section 2).

The following result is proved in ([13], Lemma 2.7).

Lemma 3.1. *Let R be a ring and I an ideal of R. If $n \geq 3$, then $E_n(R, I^2) \subset E_n(I)$.*

Lemma 3.2. *Let R be a ring and I an ideal of R. If $n \geq 3$, then $E_n(I) \subset E_n^1(R, I)$. In particular, $E_n(R, I^2) \subset E_n^1(R, I)$.*

Proof. Let $E_{ij}(x) \in E_n(I)$, where $x \in I$. If $i = 1$ or $j = 1$, then $E_{ij}(x) \in E_n^1(R, I)$. Assume $i \neq 1$ and $j \neq 1$. Then $E_{ij}(x) = E_{i1}(1)E_{1j}(x)E_{i1}(-1)E_{1j}(-x) \in E_n^1(R, I)$. $\square$

The following lemma is very crucial for later use.

Lemma 3.3. *Let A be a ring and let P be a projective A-module of rank r. Choose $s \in A$ satisfying the conditions in (2.3). Assume that if $R = A[X]/(X^2 - s^2X)$ then $E_{r+1}(R)$ acts transitively on $\text{Um}_{r+1}(R)$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P, s^2A)$.*

Proof. Without loss of generality, we may assume that A is reduced. By (2.3), there exist $p_1, \ldots, p_r \in P$ and $\phi_1, \ldots, \phi_r \in \text{Hom}(P, A)$ such that P_s is free, $(\phi_i(p_j)) =$ diagonal $(s, \ldots, s)$, $sP \subset p_1A + \cdots + p_rA$ and s is a non-zerodivisor.

Let $(a, p) \in \text{Um}(A \oplus P, s^2A)$. Replacing p by $p - ap$, we may assume that $p \in s^3P$. Since $sP \subset \sum_1^r Ap_i$, we get $p = f_1p_1 + \cdots + f_rp_r$ for some $f_i \in s^2A$. Note that $v = (a, f_1, \ldots, f_r) \in \text{Um}_{r+1}(A, s^2A)$.

Consider the following cartesian square

$$\begin{tikzcd} R \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "j_1"] \\ A \arrow[r, "j_2"'] & A/(s^2) \end{tikzcd}$$

Patching unimodular rows $(a, f_1, \ldots, f_r)$ and $(1, 0, \ldots, 0)$ over A/s^2A, we get a unimodular row $(c_0, c_1, \ldots, c_r) \in \text{Um}_{r+1}(R)$. Since $E_{r+1}(R)$ acts transitively on $\text{Um}_{r+1}(R)$, there exists $\Theta \in E_{r+1}(R)$ such that $(c_0, c_1, \ldots, c_r)\Theta = (1, 0, \ldots, 0)$. The projections of this equation gives

$$(a, f_1, \ldots, f_r)\Psi = (1, 0, \ldots, 0) \quad \text{and} \quad (1, 0, \ldots, 0)\widetilde{\Psi} = (1, 0, \ldots, 0)$$

where $\Psi, \widetilde{\Psi} \in E_{r+1}(A)$ such that $\Psi = \widetilde{\Psi}$ modulo (s^2). Hence $(a, f_1, \ldots, f_r)\Psi\widetilde{\Psi}^{-1} = (1, 0, \ldots, 0)$, where $\Psi\widetilde{\Psi}^{-1} = \Delta \in E_{r+1}(A, s^2A)$.

By (3.2), $\Delta \in E_{r+1}^1(A, sA)$. Hence by (2.5), there exists $\Theta \in E(A \oplus P)$ such that $(a, p)\Theta = (1, 0)$. This completes the proof. $\square$

As a consequence of (3.3), we prove our first result.

Theorem 3.4. *Let A be a ring of dimension d and let P be a projective A-module of rank d. Assume that if R is a finite extension of A then $E_{d+1}(R)$ acts transitively on $\text{Um}_{d+1}(R)$. Then $E(A \oplus P)$ acts transitively on $\text{Um}(A \oplus P)$.*

Proof. Let $(a, p) \in \text{Um}(A \oplus P)$. Choose $s \in A$ satisfying the conditions in (2.3). Let "bar" denote reduction modulo s^2A. Since $\dim \overline{A} = d - 1$, by Bass cancellation theorem [2], there exists $\overline{\sigma} \in E(\overline{A} \oplus \overline{P})$ such that $(\overline{a}, \overline{p})\overline{\sigma} = (1, 0)$. By (2.2), we can lift $\overline{\sigma}$ to $\theta \in E(A \oplus P)$. If $(a, p)\theta = (b, q)$, then $(b, q) \in \text{Um}(A \oplus P, s^2A)$. By (3.3), there exists $\theta_1 \in E(A \oplus P)$ such that $(b, q)\theta_1 = (a, p)\theta\theta_1 = (1, 0)$. This proves the result. $\square$

The next result generalize a result of Fasel et al. ([5], Theorem 7.3) in the case $\overline{k} = \overline{\mathbb{F}}_p$.

Theorem 3.5. *Let A be an affine algebra of dimension $d \geq 4$ over the field $\overline{\mathbb{F}}_p$, where $p \geq d$. Let P be a projective A-module of rank $d-1$. Then P is cancellative.*

Proof. By (2.7), it is enough to show that if R is any affine algebra of dimension d over $\overline{\mathbb{F}}_p$, then R^{d-1} is cancellative. Let $v \in \mathrm{Um}_d(R)$ be any unimodular row of length d. It is enough to show that there exists $\Delta \in \mathrm{GL}_d(R)$ such that $v\Delta = e_1 = (1, 0, \ldots, 0)$. Without loss of generality, we may assume that R is reduced.

Let J be the ideal of R defining the singular locus of R. Since R is reduced, height of J is ≥ 1. Let "bar" denote reduction modulo J. Then $\dim \overline{R} \leq d-1$. By (2.6), there exists $\overline{\sigma} \in E_d(\overline{R})$ such that $\overline{v}\,\overline{\sigma} = e_1$. By (2.2), we can lift $\overline{\sigma}$ to $\theta \in E_d(R)$. Then $v\theta = e_1$ modulo J. Applying (2.8), we get $\theta_1 \in \mathrm{GL}_d(R)$ such that $v\theta\theta_1 = e_1$. Hence v is completable to an invertible matrix $(\theta\theta_1)^{-1}$, i.e. R^{d-1} is cancellative. This completes the proof. □

4. Extension of Gubeladze's results

In this section we extend some results of Gubeladze. We begin by recalling three results due to Gubeladze from [6], ([7], Theorem 8.1) and ([8], Theorem 10.1) respectively. See [8] for the definition of a monoid M of Φ-simplicial growth.

Theorem 4.1. *Let M be a commutative, torsion-free, seminormal and cancellative monoid. Then for any principal ideal domain R, projective modules over $R[M]$ are free.*

Theorem 4.2. *Let R be a ring of dimension d and let $M \subset \mathbb{Q}_+^r$ be a submonoid such that $M \subset \mathbb{Q}_+^r$ is an integral extension. Then $E_n(R[M])$ acts transitively on $\mathrm{Um}_n(R[M])$ whenever $n \geq max\,(3, d+2)$.*

Theorem 4.3. *Let R be a ring of dimension d and let M be a monoid of Φ-simplicial growth. Then $E_n(R[M])$ acts transitively on $\mathrm{Um}_n(R[M])$ whenever $n \geq max\,(3, d+2)$.*

We will generalize above results of Gubeladze as follows. Since we are not assuming that M is seminormal, we need to assume that $S^{-1}P$ is free due to the following result of Gubeladze [6]: *If M is commutative, torsion-free and cancellative monoid such that projective $k[M]$-modules are free for all fields k, then M is seminormal.*

Theorem 4.4. *Let M be as in (4.2) or (4.3). Let R be a ring of dimension d and let P be a projective $R[M]$-module of rank n. Assume that $S^{-1}P$ is free, where S is the set of non-zerodivisors of R. Then $E(R[M]\oplus P)$ acts transitively on $\mathrm{Um}(R[M]\oplus P)$ whenever $n \geq max\,(2, d+1)$.*

Proof. By (2.4), we may assume that the ring $A = R[M]$ is reduced. We will use induction on d. If $d = 0$, then by assumption, projective modules of constant rank over $R[M]$ are free. Hence we are done by (4.2) and (4.3).

Assume $d > 0$. By assumption $S^{-1}P$ is free. Hence we can choose $s \in S$ such that P_s is free and conditions of (2.3) are satisfied.

Let $(a, p) \in \mathrm{Um}(A\oplus P)$ and let "bar" denote reduction modulo s^2A. Since $\dim \overline{R} = d-1$, by induction hypothesis, there exists $\phi \in E(\overline{A}\oplus\overline{P})$ such that $(\overline{a}, \overline{p})\phi = (1, 0)$. Let $\Phi \in E(A\oplus P)$ be a lift of ϕ, by (2.2). Then $(a, p)\Phi \in \mathrm{Um}(A\oplus P, s^2A)$. By Gubeladze's theorem in the free case, $E_{n+1}(B[M])$ acts transitively on $\mathrm{Um}_{n+1}(B[M])$, where $B = R[X]/(X^2 - s^2X)$ is a ring of dimension d. Applying (3.3) to $(a, p)\Phi \in \mathrm{Um}(A\oplus P, s^2A)$, there exists $\Phi_1 \in E(A\oplus P)$ such that $(a, p)\Phi\Phi_1 = (1, 0)$. This completes the proof. □

Using (4.1 and 4.4), we get the following.

Theorem 4.5. *Let M be as in (4.2) or (4.3). Further assume that M is seminormal. Let R be a ring of dimension d and let P be a projective $R[M]$-module of rank n. Then $E(R[M]\oplus P)$ acts transitively on $\mathrm{Um}(R[M]\oplus P)$ whenever $n \geq max\,(2, d+1)$.*

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